GEODESIC RESTRICTIONS OF EIGENFUNCTIONS ON ARITHMETIC SURFACES

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ABSTRACT. Let X be an arithmetic hyperbolic surface, ψ a Hecke-Maass form, and ℓ a geodesic segment on X. We obtain a power saving over the local bound of Burq-Gérard-Tzvetkov for the L^2 norm of ψ restricted to ℓ, by extending the technique of arithmetic amplification developed by Iwaniec and Sarnak.

1. Introduction

If X is a compact Riemannian manifold and ψ is a Laplace eigenfunction on X satisfying $\Delta\psi = \lambda^2\psi$, it is an interesting problem to study the extent to which ψ can concentrate on small subsets of X. Two well studied formulations of this problem are to normalise ψ by $\|\psi\|_2 = 1$, and either bound $\|\psi\|_p$ for $2 \leq p \leq \infty$ or bound the L^p norms of ψ restricted to some submanifold. We shall be interested in both of these problems in the case where X is two dimensional and the submanifold we restrict to is a geodesic segment ℓ. The basic upper bound for $\|\psi\|_p$ in this case was proven by Sogge [16] (see also Avakumović [1] and Levitan [12] when $p = \infty$), and is

$$\|\psi\|_p \ll \lambda^{\delta(p)} \tag{1}$$

where $\delta(p)$ is given by

$$\delta(p) = \begin{cases} 1/2 - 2/p & p \geq 6 \\ 1/4 - 1/2p & 2 \leq p \leq 6. \end{cases}$$

The standard bound for $\|\psi|_\ell\|_p$ is due to Burq, Gérard and Tzvetkov [7] (see also Reznikov [14]), and is

$$\|\psi|_\ell\|_p \ll \lambda^{\delta'(p)} \tag{2}$$

where $\delta'(p)$ is given by

$$\delta'(p) = \begin{cases} 1/2 - 1/p & p \geq 4 \\ 1/4 & 2 \leq p \leq 4. \end{cases}$$

Both of these bounds are sharp when X is the round 2-sphere, but can be strengthened under extra geometric assumptions on X such as negative curvature, see for instance [17, 18, 19]. It should be noted that all such improvements in the negatively curved case are by at most a power of $\log \lambda$.

We now let X be a compact arithmetic hyperbolic surface and ψ a Hecke-Maass cusp form on X, which we shall always assume to be L^2-normalised. In this case, Iwaniec and Sarnak

[11] have shown that the bound $\|\psi\|_\infty \ll \lambda^{1/2}$ given by (1) may be strengthened by a power to $\|\psi\|_\infty \ll_\epsilon \lambda^{5/12+\epsilon}$. Their approach, known as arithmetic amplification, is to construct a projection operator onto ψ using the Hecke operators as well as the wave group. It has been adapted by other authors to study the pointwise norms of arithmetic eigenfunctions in various aspects, see for instance [3, 5, 10, 20]. In this paper we apply amplification to a new kind of semiclassical problem, namely improving the exponent in the bound (2) for $\|\psi|_\ell\|_2$. Our main result is as follows.

Theorem 1. *Let ψ be a Hecke-Maass eigenfunction on X with spectral parameter t. For any geodesic segment ℓ of unit length we have*

$$\|\psi|_\ell\|_2 \ll_\epsilon t^{3/14+\epsilon}, \tag{3}$$

where the implied constant is independent of ℓ.

We may combine Theorem 1 with a theorem of Bourgain [6] to give an improvement over the local bound $\|\psi\|_4 \ll t^{1/8}$.

Corollary 2. *We have $\|\psi\|_4 \ll_\epsilon t^{1/8-1/112+\epsilon}$.*

Corollary 2 is much weaker than the bound $\|\psi\|_4 \ll_\epsilon t^\epsilon$ announced by Sarnak and Watson ([15], Theorem 3), although their result may be conditional on the Ramanujan conjecture. See also [4] for results in the case of holomorphic eigenforms. Note that Bourgain's theorem actually gives an equivalence (up to factors of t^ϵ) between a sub-local bound for $\|\psi\|_4$ and one for $\|\psi|_\ell\|_2$ that is uniform in ℓ, and so the bound of Sarnak and Watson implies Theorem 1 with an exponent of 1/8. However, we feel that our method is of interest as it does not rely on special value identities or summation formulas, and we hope to apply it to restriction problems on other groups by combining it with the techniques of [13].

Theorem 1 and the L^∞ bound of [11] can both be strengthened under the assumption that the Fourier coefficients of ψ are not small. In our case, this assumption allows us to employ an amplifier of sufficient length that it becomes profitable to estimate the Hecke recurrence using spectral methods, rather than diophantine ones as in [11]. Let $\lambda(n)$ be the normalised Hecke eigenvalues of ψ, and assume that they satisfy the bounds

$$\sum_{N \le p \le 2N} |\lambda(p)| \gg_\epsilon N^{1-\epsilon} \tag{4}$$

for all $N \ge 2$ and

$$|\lambda(n)| \le 2p^\theta \tag{5}$$

for some $\theta < 1/2$ and p prime. Note that (5) is known with $\theta = 7/64$, see [2]. We then prove

Theorem 3. *If the normalised Hecke eigenvalues $\lambda(n)$ satisfy (4) and (5), we have*

$$\|\psi|_\ell\|_2 \ll_\epsilon t^{1/(8-8\theta)+\epsilon}.$$

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2. Notation and Preliminaries

2.1. **Notation.** For simplicity, we shall restrict attention to X that arise from a quaternion division algebra $A = (\frac{a,b}{\mathbb{Q}})$ over $\mathbb{Q}$. Here $a, b \in \mathbb{Z}$ are square free and we will assume that $a > 0$. We choose a basis $1, \omega, \Omega, \omega\Omega$ for A over $\mathbb{Q}$ that satisfies $\omega^2 = a$, $\Omega^2 = b$ and $\omega\Omega + \Omega\omega = 0$. We denote the norm and trace by $N(\alpha) = \alpha\overline{\alpha}$ and $\mathrm{tr}(\alpha) = \alpha + \overline{\alpha}$. We let R be a maximal order in A, and for $m \geq 1$ let

$$R(m) = \{\alpha \in R | N(\alpha) = m\}.$$

$R(1)$ is the group of elements of norm 1; it acts on $R(m)$ by multiplication on the left and $R(1)\backslash R(m)$ is known to be finite [8]. Fix an embedding $\phi : A \to M_2(F)$, the 2×2 matrices with entries in $F = \mathbb{Q}(\sqrt{a})$ by

$$\phi(\alpha) = \begin{pmatrix} \overline{\xi} & \eta \\ b\overline{\eta} & \xi \end{pmatrix}$$

where

$$\alpha = x_0 + x_1\omega + (x_2 + x_3\omega)\Omega = \xi + \eta\Omega.$$

We define the lattice $\Gamma = \phi(R(1)) \subset SL(2, \mathbb{R})$, which is co-compact as we assumed A to be a division algebra, and let $X = \Gamma\backslash\mathbb{H}$. We define the Hecke operators $T_n : L^2(X) \to L^2(X)$, $n \geq 1$, by

$$T_n f(z) = \sum_{\alpha \in R(1)\backslash R(n)} f(\phi(\alpha)z).$$

There is a positive integer q (depending on R) such that for $(n, q) = 1$, T_n has the following properties (see [8]):

$$T_n = T_n^*, \quad \text{that is } T_n \text{ is self-adjoint,}$$
$$T_n T_m = \textstyle\sum_{d|(n,m)} dT_{nm/d^2}.$$

We let $\lambda(n)$ be the normalised Hecke eigenvalues of ψ and t be its spectral parameter, so that

$$\begin{aligned} T_n\psi &= \lambda(n)n^{1/2}\psi, \\ \Delta\psi &= (1/4 + t^2)\psi. \end{aligned}$$

We shall use the standard parametrizatons of the subgroups K, A and N of $SL_2(\mathbb{R})$, and write the Iwasawa decomposition as

$$g = n(g)\exp(h(g))k(g).$$

2.2. **Outline of the Proof.** It suffices to prove Theorem 1 with ℓ chosen to have any fixed lengt. The length we choose is the minimum of 1 and 1/100 times the injectivity radius of X, and we denote it by l. We shall prove Theorem 1 by estimating the Fourier coefficients of ψ along ℓ. More precisely, if R is the operator of restriction to ℓ and $a \in C_0^\infty(\ell)$ is a smooth cutoff function that is equal to 1 on an open subset of ℓ containing its midpoint, we shall estimate the Fourier coefficients of $aR\psi$ considered as a function on the circle $\mathbb{R}/l\mathbb{Z}$. We let x be a length co-ordinate on ℓ, and write the complex exponentials on ℓ in its identificaton with the circle as $e^{i\lambda x}$.

Using semiclassical methods, it may be shown that the contribution to the L^2 mass of $aR\psi$ coming from Fourier coefficients away from $\pm t$ is smaller than $t^{1/4}$, and so it suffices to bound the contribution from those coefficients near $\pm t$. Fortunately, this is exactly the case in which our amplification method is effective. Let β be a parameter satisfying $1 \le \beta \le t$, let $H_\beta \subset L^2(\ell)$ be the space spanned by the functions $e^{i\lambda x}$ with $\min |\lambda \pm t| \le \beta$, and let Π_β be the orthogonal projection to H_β. Using amplification, we prove the following bound for $\Pi_\beta aR\psi$.

Theorem 4. *Let $N \ge 1$ be an integer and α_n, $n \le N$, be a sequence of complex numbers. We have the bound*

$$\left| \sum_{n \le N} \alpha_n \lambda(n) \right|^2 \| \Pi_\beta aR\psi \|_2^2 \ll_\epsilon N^\epsilon t^\epsilon \left(t^{1/2} \sum_{n \le N} |\alpha_n|^2 + N t^{1/4} \beta^{1/4} \left(\sum_{n \le N} |\alpha_n| \right)^2 \right).$$

Choosing the α_n to be the amplifier used in [11] and $N = t^{1/6}\beta^{-1/6}$ gives the following corollary.

Corollary 5. *We have the bound* $\| \Pi_\beta aR\psi \|_2 \ll_\epsilon t^{5/24+\epsilon}\beta^{1/24}$.

The semiclassical bound we prove for $(1 - \Pi_\beta)aR\psi$ is as follows.

Proposition 6. *We have* $\| (1 - \Pi_\beta)aR\psi \|_2 \ll_\epsilon t^{1/4+\epsilon}\beta^{-1/4}$.

Combining these two results with $\beta = t^{1/7}$ gives Theorem 1. Note that we expect Proposition 6 to be sharp on the round sphere.

We prove Theorem 3 by improving the amplifier used in Theorem 4, and hence the bound in Corollary 5. Our new ingredient is a spectral method for estimating the number of times the Hecke operators map ℓ close to itself, which allows us to prove the following result.

Theorem 7. *Assume that ψ satisfies (4) and (5). We have the bound*

$$\| (1 - \Pi_\beta)aR\psi \|_2 \ll_\epsilon t^{\theta/2+\epsilon}\beta^{1/4-\theta/2}.$$

Theorem 3 follows by choosing $\beta = t^{(1-2\theta)/(2-2\theta)}$ and combining this with Proposition 6.

2.3. **Structure of the Paper.** We prove Proposition 6 in Section 3. We prove the unconditional amplification result, Theorem 4, in Section 4 by following closely the method of Iwaniec and Sarnak, before introducing our spectral method for estimating Hecke recurrence and proving Theorem 7 in Section 5. Sections 6 and 7 establish bounds for the oscillatory integrals which appear in the amplification argument.

3. Bounds Away from the Spectrum

We now give the proof of Proposition 6. We are free to assume that $\beta \geq t^\epsilon$, as otherwise the result follows from the bound (2) of Burq-Gérard-Tzvetkov. Let $h \in \mathcal{S}$ be a function of Payley-Wiener type that is positive, even, and ≥ 1 in the interval $[-1, 1]$. Let $h_t(\lambda) = h(\lambda - t) + h(-\lambda - t)$, and let k_t^0 be the K-biinvariant function on $\mathbb{H}$ with Harisch-Chandra transform h_t. It is of compact support which may be chosen arbitrarily small, by the Payley-Wiener theorem of Gangolli [9]. We shall need the following results about k_t, which follow easily from the methods developed in [13].

Proposition 8. *Let $p_t \in C_0^\infty(\mathbb{R})$ be the function such that $k_t(a(x)) = p_t(x)$. We have the bound*

$$|p_t(x)| \ll t(1 + tx)^{-1/2}, \tag{6}$$

and the asymptotic

$$p_t(x) = e^{itx} a(x) t^{1/2} x^{-1/2} + e^{itx} b(x) t^{1/2} x^{-1/2} + tO((tx)^{-3/2}) \tag{7}$$

for $a(x)$ and $b(x)$ in $C_0^\infty(\mathbb{R})$.

Proof. Inequality (6) follows from Theorem 3 of [13] and inversion of the Harish-Chandra transform. To prove (7), it follows from Propositions 13 and 14 of [13] that $p_t(x)$ has an expression of the form

$$p_t(x) = te^{itx} \int_{-\infty}^{\infty} h_1(x, z) e^{itxz^2} dz + te^{-itx} \int_{-\infty}^{\infty} h_2(x, z) e^{itxz^2} dz + tO_A((tx)^{-A}),$$

where h_1 and h_2 are smooth and compactly supported. The asymptotic follows immediately from this by stationary phase.

$\square$

Let A_t^0 denote the operator on X obtained by averaging the point pair invariant associated to k_t^0 under Γ, and let $R_t = RA_t^0$. If $\phi \in H_\beta^\perp$ has norm 1, it suffices to estimate the pairing $\langle aR_t\psi, \phi \rangle$. We take adjoints and apply Cauchy's inequality to obtain

$$\begin{aligned} |\langle aR_t\psi, \phi\rangle| &= |\langle \psi, R_t^* a\phi\rangle| \\ &\leq \langle a\phi, R_t R_t^* a\phi\rangle^{1/2}. \end{aligned}$$

Let k_t be the K-biinvariant function with Harish-Chandra transform h_t^2, and let $K_t(x, y)$ be the associated point pair invariant on $\mathbb{H}$. If we choose the support of k_t^0 to be sufficiently small, the integral kernel of $R_t R_t^*$ is just the restriction of K_t to $\ell \times \ell$. Therefore, if we let $\rho : [0, l] \to \ell$ be a parametrization of ℓ and $p_t \in C_0^\infty(\mathbb{R})$ be the function defined in the statement of Proposition 8, we have $K_t(\rho(x), \rho(y)) = p_t(x - y)$. Recalling our identification of ℓ with $\mathbb{R}/l\mathbb{Z}$, we define $P_t : L^2(\ell) \to L^2(\ell)$ to be the operator with kernel

$$P_t(\rho(x), \rho(y)) = \sum_{n \in \mathbb{Z}} p_t(x - y + nl).$$

If we choose the cutoff function a so that its support is bounded away from the ends of ℓ and the support of k_t^0 to be sufficiently small, we will have $R_t R_t^* a\phi_1 = P_t a\phi_1$ so that it suffices to estimate $\langle a\phi, P_t a\phi\rangle$. Define I_β to be the union of the intervals $[\pm t - \beta/2, \pm t + \beta/2]$, and decompose $a\phi$ as $\phi_1 + \phi_2$, where the Fourier transform of ϕ_2 is supported on I_β and the transform of ϕ_1 is supported on its complement. Because a was a fixed smooth function, we have $\|\phi_2\|_2 \ll_A \beta^{-A}$. Because the kernel of P_t is translation invariant, we have

$$\begin{aligned}\langle a\phi, P_t a\phi\rangle &= \langle \phi_1, P_t\phi_1\rangle + \langle \phi_2, P_t\phi_2\rangle \\ &\leq \sup_{\lambda\notin I_\beta} |\widehat{p_t}(\lambda)| + O_A(\beta^{-A}) \sup |\widehat{p_t}(\lambda)|.\end{aligned}$$

Inequality (6) of Proposition 8 gives $\sup |\widehat{p_t}(\lambda)| \ll t^{1/2}$, and so Proposition 6 will follow from the following estimate.

Lemma 9. *We have $|\widehat{p_t}(\lambda)| \ll_\epsilon t^{1/2+\epsilon}\beta^{-1/2}$ for $\lambda \notin I_\beta$.*

Proof. We wish to estimate the integral

$$\int_{-\infty}^{\infty} p_t(x) e^{i\lambda x} dx$$

for $\lambda \notin I_\beta$. We let $b(x)$ be a smooth cutoff function that is equal to 1 on $[-1,1]$ and zero outside $[-2,2]$, and decompose the integral as

$$\int_{-\infty}^{\infty} p_t(x) e^{i\lambda x} dx = \int_{-\infty}^{\infty} b(t^{-\epsilon}\beta x) p_t(x) e^{i\lambda x} dx + \int_{-\infty}^{\infty} (1 - b(t^{-\epsilon}\beta x)) p_t(x) e^{i\lambda x} dx.$$

The contribution from the first integral may be bounded by $t^{1/2+\epsilon}\beta^{-1/2}$ using (6) of Proposition 8, and we shall use the asymptotic (7) and integration by parts to show that the second integral is small.

To bound the contribution of the error term $tO((tx)^{-3/2})$ in the asymptotic for $p_t(x)$, we observe that

$$\int_{t^{-1}}^{1} (xt)^{-3/2} dx \ll t^{-1}$$

and so this term makes a contribution of $O(1)$ which may be ignored. The two main terms in the asymptotic are identical, and so we shall treat the first one by estimating the integral

$$\int_{-\infty}^{\infty} (1 - b(t^{-\epsilon}\beta x)) e^{i(\lambda - t)x} a(x) t^{1/2} x^{-1/2} dx.$$

After changing variable from x to $t^{-\epsilon}\beta x$, this becomes

$$t^{1/2+\epsilon}\beta^{-1/2} \int_{-\infty}^{\infty} (1 - b(x)) e^{i(\lambda - t)t^\epsilon \beta^{-1} x} a(t^\epsilon \beta^{-1} x) x^{-1/2} dx.$$

Our assumption that $\beta \geq t^\epsilon$ for some ϵ implies that $t^\epsilon\beta^{-1} \leq 1$, so that all derivatives of $a(t^\epsilon\beta^{-1}x)$ are bounded. In addition, all derivatives of $x^{-1/2}$ are bounded on the support of $(1 - b(x))$, and $(\lambda - t)t^\epsilon\beta^{-1} \gg t^\epsilon$, so repeated integration by parts implies that this integral is $\ll_A t^{-A}$ as required.

$\square$

4. Arithmetic Amplification I

We now prove Theorem 4 using the method of arithmetic amplification developed by Iwaniec and Sarnak in [11]. Let $N \geq 1$ be an integer, and let α_n, $n \leq N$, be a sequence of complex numbers. As in Section 3 we let A_t^0 denote the spectral projector on X associated to k_t^0, and $R_t = RA_t^0$. We define $\mathcal{T}$ to be the operator

$$\mathcal{T} = \sum_{1 \leq n \leq N} \frac{\alpha_n}{\sqrt{n}} T_n,$$

and will estimate the inner product $\langle aR_t \mathcal{T}\psi, \phi \rangle$ with $\phi \in H_\beta$ an arbitrary norm one vector. We begin by taking adjoints and applying Cauchy-Schwarz as follows:

$$\begin{aligned} |\langle aR_t\mathcal{T}\psi, \phi\rangle| &= |\langle \psi, \mathcal{T}^* R_t^* a\phi\rangle| \\ &\leq \langle \mathcal{T}^* R_t^* a\phi, \mathcal{T}^* R_t^* a\phi\rangle^{1/2} \\ &= \langle a\phi, \mathcal{T}\mathcal{T}^* R_t R_t^* a\phi\rangle^{1/2}. \end{aligned}$$

As in Section 3, we let k_t be the K-biinvariant function with Harish-Chandra transform h_t^2, and $K_t(x,y)$ the associated point pair invariant. By abuse of notation, we shall also let $K_t(x,y)$ denote the symmetrised integral kernel on X. We then have

$$|\langle aR_t\mathcal{T}\psi, \phi\rangle| \leq \langle a\phi, \mathcal{T}\mathcal{T}^* K_t a\phi\rangle^{1/2}. \tag{8}$$

We shall estimate this inner product by estimating the terms $\langle a\phi, T_n K_t a\phi\rangle$. If we define $I(t, \ell, \gamma\ell)$ by

$$I(t,\ell,\gamma\ell) = \int_\ell \int_{\gamma\ell} a(x_1)\phi(x_1)a(x_2)\overline{\phi(x_2)} K_t(x_1,x_2) dx_1 dx_2,$$

then the RHS of (8) may be expressed as

$$\langle a\phi, \mathcal{T}\mathcal{T}^* K_t a\phi\rangle = \sum_{m,n\leq N} \alpha_n \overline{\alpha}_m \sum_{d|(n,m)} \frac{d}{\sqrt{mn}} \sum_{\gamma \in R(nm/d^2)} I(t,\ell,\gamma\ell). \tag{9}$$

We shall estimate (9) with the aid of a distance function on pairs of geodesics on X. If ℓ' is another geodesic segment on X of the same length as ℓ, we define the distance $d(\ell, \ell')$ between the two segments as follows. Extend ℓ by twice its length in both directions, and let $N_\ell(r)$ be the set of all points within distance r of this extended geodesic. We define

$$d(\ell,\ell') = \inf\{r | \ell' \subseteq N_\ell(r)\}.$$

The first result we shall need to prove Theorem 5 says that only those γ for which $d(\ell, \gamma\ell) \ll t^{-1/2+\delta}$ make a large contribution to the sum in (9).

Proposition 10. *Let ℓ and ℓ' be two geodesic segments of length l, and let $\kappa = d(\ell,\ell')$. We have the estimate*

$$|I(t,\ell,\ell')| \ll t^{1/2} \tag{10}$$

for all κ, while if $\kappa \geq t^{-1/2+\epsilon}\beta^{1/2}$ we have

$$|I(t,\ell,\ell')| \ll_\epsilon t^{1/2+\epsilon}\max\{(\kappa t^{1/2})^{-1}, \beta^2(\kappa t^{1/2})^{-3}\}. \tag{11}$$

We shall prove Proposition 10 in Section 5. The second result we shall need is a bound for the counting function

$$M(\kappa) = M(\ell, n, \kappa) = |\{\gamma \in R(n) | d(\gamma\ell, \ell) < \kappa\}|.$$

Lemma 11.

$$M(\ell, n, \kappa) \ll_\epsilon (\kappa^2 + \kappa^{1/2})n^{1+\epsilon} + n^\epsilon.$$

Proof. This may be proven in exactly the same way as the corresponding Lemma 1.3 of [11]. The only differences are that we must consider the quadratic form $[\alpha, \beta, \gamma]$ associated to ℓ with

$$\beta^2 - 4\alpha\gamma = 1,$$

and the subgroup K_ℓ generated by translation along ℓ which may be parametrized as

$$K_\ell = \left\{ \begin{bmatrix} t-\beta u & -2\gamma u \\ 2\alpha u & t+\beta u \end{bmatrix} | t^2 - u^2 = 1 \right\}.$$

We have

$$d(\ell, \gamma\ell) < \kappa \to \gamma = z + O(\kappa) \quad \text{with} \quad z \in K_\ell.$$

If we write γ as

$$\gamma = \frac{1}{\sqrt{n}} \begin{bmatrix} x_0 - x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ bx_2 - bx_3\sqrt{a} & x_0 + x_1\sqrt{a} \end{bmatrix}$$

then x_0 and x_1 must satisfy the equations

$$\left|x_0^2 - \frac{a}{\beta^2}x_1^2 - n\right| \ll n\kappa, \quad |x_0| \ll \sqrt{n}, \quad |x_1| \ll \sqrt{n},$$

where the last two conditions come from the fact that the entries of γ must be bounded. The proof now proceeds exactly as in [11], with the difference that we must count ideals of a given norm in real quadratic fields rather than imaginary ones, and the presence of units intorduces an extra factor of n^ϵ into our counting which we may ignore.

$\square$

With these results, we are ready to estimate the sum (9). We begin with a dyadic decomposition, by defining $I_0 = [0, t^{-1/2+\epsilon}\beta^{1/2}]$ and $I_k = [2^{k-1}t^{-1/2+\epsilon}\beta^{1/2}, 2^k t^{-1/2+\epsilon}\beta^{1/2}]$ for $\log t \gg k \geq 1$, and break the sum into pieces with $d(\ell, \gamma\ell) \in I_k$. When $k = 0$, we apply the bounds

$$\begin{aligned} |I(t, \ell, \gamma\ell)| &\ll t^{1/2} \\ M(\ell, nm/d^2, t^{-1/2+\epsilon}\beta^{1/2}) &\ll_\epsilon t^{-1/4+\epsilon}\beta^{1/4}(nm/d^2)^{1+\epsilon} + (nm/d^2)^\epsilon \end{aligned}$$

from Lemmas 10 and 11 to give the following estimate for the sum.

$$\sum_{m,n\le N}\alpha_n\overline{\alpha}_m\sum_{d|(n,m)}\frac{d}{\sqrt{mn}}\sum_{\substack{\gamma\in R(nm/d^2),\\ d(\ell,\gamma\ell)\in I_0}} I(t,\ell,\gamma\ell) \ll N^\epsilon t^\epsilon\sum_{m,n\le N}\alpha_n\overline{\alpha}_m\sum_{d|(n,m)}\frac{d}{\sqrt{mn}}\left(t^{1/4}\beta^{1/4}\frac{nm}{d^2}+t^{1/2}\right)$$

$$\ll N^\epsilon t^\epsilon\sum_{m,n\le N}\alpha_n\overline{\alpha}_m\sum_{d|(n,m)}\frac{\sqrt{mn}}{d}t^{1/4}\beta^{1/4}+\frac{d}{\sqrt{mn}}t^{1/2}. \tag{12}$$

When $k\ge 1$, we apply the bounds

$$\begin{aligned}|I(t,\ell,\gamma\ell)| &\ll 2^{-k}t^{1/2+\epsilon}\beta^{-1/2}\\ M(\ell,nm/d^2,t^{-1/2+\epsilon}2^k) &\ll_\epsilon 2^{k/2}t^{-1/4+\epsilon/2}\beta^{1/4}(nm/d^2)^{1+\epsilon}+(nm/d^2)^\epsilon\end{aligned}$$

to obtain

$$\sum_{m,n\le N}\alpha_n\overline{\alpha}_m\sum_{d|(n,m)}\frac{d}{\sqrt{mn}}\sum_{\substack{\gamma\in R(nm/d^2),\\ d(\ell,\gamma\ell)\in I_k}} I(t,\ell,\gamma\ell) \tag{13}$$

$$\ll N^\epsilon t^\epsilon\sum_{m,n\le N}\alpha_n\overline{\alpha}_m\sum_{d|(n,m)}\frac{\sqrt{mn}}{d}2^{-k/2}t^{1/4}\beta^{-1/4}+\frac{d}{\sqrt{mn}}2^{-k}t^{1/2}\beta^{-1/2}.$$

The RHS of (13) is at most $2^{-k/2}$ times the RHS of (12), and so the sum over k is at most a constant times the RHS of (12). We have

$$\sum_{m,n\le N}\sum_{d|(n,m)}\frac{\sqrt{nm}}{d}|\alpha_n\alpha_m|\le N^{1+\epsilon}\left(\sum_{n\le N}|\alpha_n|\right)^2, \tag{14}$$

while

$$\begin{aligned}\sum_{m,n\le N}\sum_{d|(m,n)}\frac{d}{\sqrt{mn}}|\alpha_n\alpha_m| &= \sum_{\substack{m\le N\\ n\le N\\ (m,n)=1}}\sum_{\substack{lm\le N\\ ln\le N}}\sum_{d|l}\frac{d}{l\sqrt{mn}}|\alpha_{nl}\alpha_{ml}|\\ &\le N^\epsilon\sum_{\substack{lm\le N\\ ln\le N}}\left(\frac{|\alpha_{nl}|^2}{m}+\frac{|\alpha_{ml}|^2}{n}\right)\\ &\ll N^\epsilon\sum_{n\le N}|\alpha_n|^2.\end{aligned} \tag{15}$$

Combining (14) and (15) with (12) gives

$$\langle a\phi,\mathcal{T}\mathcal{T}^*K_ta\phi\rangle\ll N^\epsilon t^\epsilon\left(t^{1/2}\sum_{n\le N}|\alpha_n|^2+Nt^{1/4}\beta^{1/4}\left(\sum_{n\le N}|\alpha_n|\right)^2\right),$$

and Theorem 4 now follows by applying (8) and calculating the action of $\mathcal{T}$ on ψ.

5. Arithmetic Amplification II

We now present the modifications which we are able to make to the proof of Theorem 4 under the assumptions (4) and (5), which result in the proof of Theorem 7. Let N be an integer of size roughly $t^{1/2+\epsilon}\beta^{-1/2}$ for some $\epsilon > 0$, and define $\mathcal{T}$ to be the operator

$$\mathcal{T} = \sum_{N/2<p<N} \frac{\lambda(p)}{\sqrt{p}} T_p.$$

If K_t is as in Section 3, it again suffices to bound the inner product $\langle a\phi, \mathcal{T}\mathcal{T}^* K_t a\phi \rangle$. After reducing $\mathcal{T}\mathcal{T}^*$ using the Hecke relations, we have

$$\langle a\phi, \mathcal{T}\mathcal{T}^* K_t a\phi \rangle = \sum_{N/2<p<N} I(t,\ell,\ell) + \sum_{N/2<p,q<N} \lambda(p)\overline{\lambda(q)} \frac{1}{\sqrt{pq}} \sum_{\gamma \in R(pq)} I(t,\ell,\gamma\ell). \tag{16}$$

The key difference between the proof of Theorem 4 and Theorem 7 is that we shall now estimate the recurrences of ℓ under a large collection of Hecke operators T_n at once using spectral methods, rather than individually. This is carried out in the following proposition.

Proposition 12. *If M and $\delta > 0$ satisfy $M \geq \delta^{-2-\epsilon}$, we have*

$$\sum_{\substack{M/2<m<M \\ (m,q)=1}} \sum_{\substack{\gamma \in R(m) \\ d(\ell,\gamma\ell) \leq \delta}} \frac{1}{\sqrt{m}} \ll_\epsilon \delta^2 M^{3/2},$$

where q is the integer defined in Section 2.1.

Proof. We may assume without loss of generality that δ is less than any fixed constant. Let $\tilde{\ell}$ be the lift of ℓ to $\Gamma\backslash SL_2(\mathbb{R})$, and fix a Riemannian metric on a neighbourhood of $\tilde{\ell}$. Let f be a smooth positive bump function which is supported on a tube of radius 2δ around $\tilde{\ell}$ and equal to $C(\delta)\delta^{-1}$ on a tube of radius δ, where the tubes are constructed using the Riemannian metric we have chosen. $C(\delta)$ is some constant which is absolutely bounded away from 0 and ∞, and chosen so that f satisfies $\|f\|_2 = 1$ in $L^2(\Gamma\backslash SL_2(\mathbb{R}))$ and $\langle f, \gamma f\rangle \gg 1$ if $d(\ell,\gamma\ell) \leq \delta$.

Choose $g \in C_0^\infty(0,\infty)$ to be positive and satisfy $g(x) = 1$ for $1/2 \leq x \leq 1$. If we define

$$\mathcal{S} = \sum_{(m,q)=1} \frac{g(m/M)}{\sqrt{m}} T_m,$$

then we have

$$\sum_{\substack{M/2<m<M \\ (m,q)=1}} \sum_{\substack{\gamma \in R(m) \\ d(\ell,\gamma\ell) \leq \delta}} \frac{1}{\sqrt{m}} \ll \langle f, \mathcal{S} f\rangle$$

and we may estimate the RHS spectrally. Expand f with respect to a decomposition of $L^2(\Gamma\backslash SL_2(\mathbb{R}))$ into automorphic representations as

$$f = \sum_i \alpha_i \psi_i,$$

where ψ_i is an L^2 normalised vector in an automorphic representation with eigenvalue μ_i under the Casimir operator C. We may choose f so that

$$\| C^n f \|_2 \ll_n \delta^{-1-2n}.$$

Integration by parts then gives

$$\begin{aligned} \langle f, \psi_i \rangle &= \mu_i^{-n} \langle f, C^n \psi_i \rangle \\ &= \mu_i^{-n} \langle C^n f, \psi_i \rangle \\ &\ll_n |\mu_i|^{-n} \delta^{-1-2n}, \end{aligned}$$

so that $|\alpha_i| \ll_{A,\epsilon} \delta^A$ if $|\mu_i| > \delta^{-2-\epsilon}$, and we may write

$$f = \langle f, 1 \rangle + \sum_{|\mu_i| \leq \delta^{-2-\epsilon}}{}' \alpha_i \psi_i + O_{A,\epsilon}(\delta^A).$$

Note that we have normalised the volume of $\Gamma \backslash SL_2(\mathbb{R})$ to be 1, and Σ' denotes the sum over the nontrivial representations. Substituting this into $\langle \mathcal{S} f, f \rangle$ gives

$$\langle \mathcal{S} f, f \rangle = \langle f, 1 \rangle^2 \sum_{(m,q)=1} g(m/M) \sqrt{m} + \sum_{|\mu_i| \leq \delta^{-2-\epsilon}} |\alpha_i|^2 \sum_{(m,q)=1} g(m/M) \lambda_i(m) + O_{A,\epsilon}(\delta^A),$$

where $\lambda_i(m)$ are the Hecke eigenvalues of ψ_i. The result now follows from Lemma 13 below, and the asymptotic $\langle f, 1 \rangle \ll \delta$. (Note that our assumptions that $M \geq \delta^{-2-\epsilon}$ and $|\mu_i| \leq \delta^{-2-\epsilon'}$ guarantee that the hypothesis of the lemma is satisfied.)

□

Lemma 13. *If $M \geq |\mu_i|^{1+\epsilon}$, we have*

$$\left| \sum_m g(m/M) \lambda_i(m) \right| \ll_{A,\epsilon} M^{-A},$$

where the implied constant is uniform in ψ_i.

Proof. We shall drop the subscript i, and assume that ψ is a vector in a principal series representation as the discrete series case is similar. We first consider the case $q = 1$.

Let r be the spectral parameter of ψ, so that $\mu = 1/4 + r^2$. By applying the functional equation and Stirling's formula, we see that the L-function $L(s, \psi)$ satisfies the estimate

$$|L(-A + it, \psi)| \ll_{A,\epsilon} (t^2 + r^2 + 1)^{A+1/2+\epsilon} \tag{17}$$

for A sufficiently large. If we let $\widehat{g}(s)$ be the Mellin transform of g, which is entire and decays rapidly in vertical strips, we obtain

$$\sum_m g(m/M) \lambda(m) = \int_{(2)} L(s, \psi) \widehat{g}(s) M^s ds.$$

If we shift the line of integration to $\sigma = -A$, and apply (17) and the rapid decay of $\widehat{g}$, we have

$$\begin{aligned} \left|\sum_m g(m/M)\lambda_i(m)\right| &\ll_{\epsilon'} M^{-A}(1+r^2)^{A+1/2+\epsilon'} \\ &\ll_{\epsilon'} M^{-A}\mu^{A+1/2+\epsilon'} \\ &\ll_{\epsilon'} M^{-A}M^{(1-\epsilon)(A+1/2+\epsilon')} \\ &\ll_{B,\epsilon} M^{-B} \end{aligned}$$

as required. In the case when $q > 1$, we apply the same argument to the incomplete L-function obtained by removing the local factors at primes dividing q from $L(s,\psi)$.

$\square$

With these results, we are ready to estimate the RHS of (16). We begin by applying the trivial bound of Proposition 10 to the first sum, and our assumption that $|\lambda(p)| \leq 2p^\theta$ to the second, which gives

$$\langle a\phi, \mathcal{T}\mathcal{T}^* K_t a\phi\rangle \ll Nt^{1/2} + N^{2\theta} \sum_{p,q} \frac{1}{\sqrt{pq}} \sum_{\gamma \in R(pq)} |I(t,\ell,\gamma\ell)|.$$

Enlarging the sum to one over all $N^2/4 < n < N^2$ with $(n,q) = 1$ gives

$$\langle a\phi, \mathcal{T}\mathcal{T}^* K_t a\phi\rangle \ll Nt^{1/2} + N^{2\theta} \sum_{\substack{n\sim N^2 \\ (n,q)=1}} \frac{1}{\sqrt{n}} \sum_{\gamma \in R(n)} |I(t,\ell,\gamma\ell)|. \tag{18}$$

We now apply a dyadic decomposition with respect to $d(\ell,\gamma\ell)$. Define $I_0 = [0, t^{-1/2+\epsilon}\beta^{1/2}]$ and $I_k = [2^{k-1}t^{-1/2+\epsilon}\beta^{1/2}, 2^k t^{-1/2+\epsilon}\beta^{1/2}]$ for $\log t \gg k \geq 1$, and decompose the sum in (18) as

$$\begin{aligned} \sum_{\substack{n\sim N^2 \\ (n,q)=1}} \frac{1}{\sqrt{n}} \sum_{\gamma \in R(n)} |I(t,\ell,\gamma\ell)| &= \sum_k \sum_{\substack{n\sim N^2 \\ (n,q)=1}} \sum_{\substack{\gamma\in R(n) \\ d(\ell,\gamma\ell)\in I_k}} \frac{1}{\sqrt{n}} |I(t,\ell,\gamma\ell)| \\ &\leq \sum_k \sup_{d(\ell,\gamma\ell)\in I_k} |I(t,\ell,\gamma\ell)| \sum_{\substack{n\sim N^2 \\ (n,q)=1}} \sum_{\substack{\gamma\in R(n) \\ d(\ell,\gamma\ell)\in I_k}} \frac{1}{\sqrt{n}} \end{aligned}$$

The assumption that $N \sim t^{1/2+\epsilon}\beta^{-1/2}$ implies that we may choose $\delta = 2^k t^{-1/2+\epsilon}\beta^{1/2}$ and $M = N^2$ in Proposition 12, so that

$$\sum_{\substack{n\sim N^2 \\ (n,q)=1}} \sum_{\substack{\gamma\in R(n) \\ d(\ell,\gamma\ell)\in I_k}} \frac{1}{\sqrt{n}} \ll N^3 2^{2k} t^{-1+\epsilon}\beta.$$

When $k = 0$, combining this with the bound $|I(t,\ell,\gamma\ell)| \ll t^{1/2}$ from Lemma 10 gives

$$\sum_{\substack{n\sim N^2\\(n,q)=1}}\sum_{\substack{\gamma\in R(n)\\ d(\ell,\gamma\ell)\in I_0}}\frac{1}{\sqrt{n}}|I(t,\ell,\gamma\ell)|\ll_\epsilon N^3t^{-1/2+\epsilon}\beta.$$

When $k\geq 1$ but $2^k\leq\beta^{1/2}$, we choose the first term in the minimum of inequality (11) to obtain

$$|I(t,\ell,\gamma\ell)|\ll 2^{-k}t^{1/2+\epsilon}\beta^{-1/2},$$

so that

$$\sum_{\substack{n\sim N^2\\(n,q)=1}}\sum_{\substack{\gamma\in R(n)\\ d(\ell,\gamma\ell)\in I_k}}\frac{1}{\sqrt{n}}|I(t,\ell,\gamma\ell)|\ll_\epsilon N^3 2^k t^{-1/2+\epsilon}\beta^{1/2}.$$

When $2^k>\beta^{1/2}$, choosing the second term in inequality (11) yields

$$|I(t,\ell,\gamma\ell)|\ll 2^{-3k}t^{1/2+\epsilon}\beta^{1/2}$$

and

$$\sum_{\substack{n\sim N^2\\(n,q)=1}}\sum_{\substack{\gamma\in R(n)\\ d(\ell,\gamma\ell)\in I_k}}\frac{1}{\sqrt{n}}|I(t,\ell,\gamma\ell)|\ll_\epsilon N^3 2^{-k} t^{-1/2+\epsilon}\beta^{3/2}.$$

Summing over k gives

$$\sum_{\substack{n\sim N^2\\(n,q)=1}}\frac{1}{\sqrt{n}}\sum_{\gamma\in R(n)}|I(t,\ell,\gamma\ell)|\ll_\epsilon N^3t^{-1/2+\epsilon}\beta,$$

and substituting back into (18) we have

$$\langle a\phi,\mathcal{T}\mathcal{T}^*K_t a\phi\rangle\ll Nt^{1/2}+N^{3+2\theta}t^{-1/2+\epsilon}\beta.$$

Substituting this into our Cauchy-Schwarz inequality, this implies the bound

$$|\langle aR_t\mathcal{T}\psi,\phi\rangle|^2\ll Nt^{1/2}+N^{3+2\theta}t^{-1/2+\epsilon}\beta.$$

If we estimate the action of $\mathcal{T}$ on ψ using our assumption (4) and substitute $N\sim t^{1/2+\epsilon}\beta^{-1/2}$, this gives

$$|\langle aR_t\mathcal{T}\psi,\phi\rangle|\ll_\epsilon t^{\theta/2+\epsilon}\beta^{1/4-\theta/2}$$

as required.

Remark. The method we have used of estimating Hecke recurrences spectrally is unlikely to work in other situations. It requires us to choose an amplifier that makes the sums of eigenvalues in Proposition 12 longer than the relevant analytic conductors, and in other cases (such as higher rank or when using the operators T_{p^2} on GL_2 to give an unconditional theorem) this gives the amplifier so much mass that the 'off-diagonal' term is worse than the trivial bound. The method also depends on the exponent of κ in Proposition 10 being

small, and fails to improve the L^∞ bound of [11] under the assumption (4) because the corresponding exponent in that case is larger.

6. Oscillatory Integrals

We now establish a bound for integrals of spherical functions over geodesic segments, stated as Proposition 14 below, that will be needed in proof of Proposition 10.

Let t and κ be as in Proposition 10, and note that because $l \leq 1$, we may assume that $\kappa \leq 2$ by choosing the support of k_t to be small. Let w be a parameter satisfying $t^{-1+\epsilon}\kappa^{-2} \leq w \leq 10$, and let $b \in C_0^\infty(\mathbb{R})$ be a cutoff function around the origin at scale w, i.e. such that

$$\operatorname{supp} b(z) \quad \subseteq \quad [-10w, 10w], \tag{19}$$

$$\left| \frac{d^n}{dz^n} b(z) \right| \quad \ll_n \quad w^{-n}. \tag{20}$$

Let x be a number satisfying $C^{-1}\kappa w \leq x \leq C\kappa w$ for some absolute constant $C > 0$, and let $\rho(y) = x + ie^y = n(x)a(y)$ be a function parametrizing an infinite vertical geodesic in $\mathbb{H}$. Let s satisfy $|t - s| \leq t^\epsilon$, and let φ_{-s} be the spherical function with parameter $-s$. The result we shall use is as follows.

Proposition 14. *With notations as above, if $\lambda \in [\pm t - \beta, \pm t + \beta]$ we have*

$$\left| \int_{-\infty}^{\infty} b(y) e^{i\lambda y} \varphi_{-s}(\rho(y)) dy \right| \ll_A t^{-A}. \tag{21}$$

Proof. We may assume without loss of generality that $\lambda \in [t - \beta, t + \beta]$, as the proof in the other case is identical. If we substitute the formula for φ_{-s} as an integral of plane waves into the LHS of (21), it becomes

$$\int_{-\infty}^{\infty} \int_{-\pi}^{\pi} b(y) \exp(i\lambda y + (1/2 - is) h(k(\theta) n(x) a(y))) d\theta dy.$$

We first make a smooth decomposition of the integral in θ, into the piece with $|\theta| \leq \kappa$ and the complement. Let b_1 and b_2 be a smooth partition of unity on $[-\pi, \pi]$ with b_1 supported outside $(-c\kappa, c\kappa)$ and b_2 supported in $[-2c\kappa, 2c\kappa]$, where c is a constant to be chosen later.

6.1. Contribution from θ away from 0.

We shall prove the rapid decay of the first integral

$$\int_{-\infty}^{\infty} \int_{-\pi}^{\pi} b(y) b_1(\theta) \exp(i\lambda x + (1/2 - is) h(k(\theta) n(x) a(y))) d\theta dy$$

by establishing it for the sub-integral

$$\int_{-\infty}^{\infty} b(y) \exp(i\lambda y - ish(k(\theta) n(x) a(y))) dy, \tag{22}$$

where we have absorbed the factor of $\exp(h(k(\theta)n(x)a(y))/2)$ into $b(y)$. The support of b_1 allows us to assume that $|\theta| \geq c\kappa$. We shall need the following lemma on the function $h(k(\theta)n(x)a(y))$.

Lemma 15. *If (x, y) is restricted to a bounded open set $I \subset \mathbb{R}^2$, there exists an open neighbourhood $U \subset K$ of the identity and a nonvanishing real analytic function z on $U \times I$ such that*

$$(\partial/\partial y)h(k(\theta)n(x)a(y)) = 1 - \theta^2 z$$

for $(x, y) \in I$ and $k(\theta) \in U$.

Proof. Let α be the angle between the geodesic $k(\theta)\ell$ and the upward pointing vector at the point $k(\theta)n(x)a(y)$, which has the property that

$$k(k(\theta)n(x)a(y)) = k(\alpha).$$

The analyticity of the Iwasawa decomposition implies that α is analytic as a function of (θ, x, y). We have

$$\begin{aligned} 1 - (\partial/\partial y)h(k(\theta)n(x)a(y)) &= 1 - \cos\alpha \\ &= -2\sin^2(\alpha/2). \end{aligned}$$

We choose the neighbourhood U such that $\sin(\alpha/2)$ vanishes on $U \times I$ iff $\theta = 0$. It can be seen that $\partial\alpha/\partial\theta$ never vanishes when $\theta = 0$, and so because α was analytic we see that there is a nonvanishing analytic function z_0 on $U \times I$ such that $\sin(\alpha/2) = \theta z_0$. Defining $z = 2z_0^2$ gives the result.

□

Apply Lemma 15 to any set I that contains $[0, 10C\kappa] \times [-100, 100]$ (which contains all (x, y) under consideration), and let $U \subset K$ be the set produced by the Lemma. In the case when $\theta \in U$, we may apply Lemma 15 and take an antiderivative to obtain

$$h(k(\theta)n(x)a(y)) = y - \theta^2 Z + c(\theta, x),$$

where Z is an antiderivative of z with respect to y. We may use this to rewrite the integral (22) as

$$\begin{aligned} \int_{-\infty}^{\infty} b(y)\exp(i\lambda y - ish(k(\theta)n(x)a(y)))dy &= e^{ic(\theta,x)} \int_{-\infty}^{\infty} b(y)\exp(i(\lambda - s)y - is\theta^2 Z)dy \\ &= e^{ic(\theta,x)} \int_{-\infty}^{\infty} b(y)\exp(-is\theta^2\Psi)dy, \end{aligned} \tag{23}$$

where we define $\Psi = Z + s^{-1}\theta^{-2}(s - \lambda)y$.

We have assumed that $|t - \lambda| \leq \beta$ and $|t - s| \leq t^\epsilon$, and we may further assume without loss of generality that $\beta \geq t^\epsilon$ so that $|s - \lambda| \leq 2\beta$. Combined with our assumption that $\theta \geq c\kappa \geq ct^{-1/2+\epsilon}\beta^{1/2}$, this implies that

$$|s^{-1}\theta^{-2}(s - \lambda)| \ll s^{-1}t^{1-\epsilon}\beta^{-1}2\beta \ll t^{-\epsilon},$$

so that

$$\Psi = Z + O(t^{-\epsilon})y, \quad \text{and} \quad (\partial/\partial y)\Psi = z + O(t^{-\epsilon}). \tag{24}$$

It follows from (24) and the fact that z was nonvanishing that $(\partial/\partial y)\Psi$ will be uniformly bounded from below on $U \times I$ for t sufficiently large (after shrinking the domain if necessary). This implies that all derivatives of y with respect to Ψ are uniformly bounded from above, so we may change variable in (23) and write it as

$$\int_{-\infty}^{\infty} b(\Psi) e^{-is\theta^2\Psi} d\Psi$$

where b is again a cutoff function at scale w. We therefore see that the integral is $\ll_A t^{-A}$ if $s\theta^2 w \gg t^\epsilon$, but this follows from our assumptions that $|\theta| \geq c\kappa$ and $w \geq \kappa^{-2}t^{-1+\epsilon}$.

In the case where θ is outside of U, we may show that $(\partial/\partial y)h(k(\theta)n(x)a(y)) \leq 1 - c_1$ for some $c_1 > 0$ depending only on U, which gives

$$(\partial/\partial y)(i\lambda s^{-1}y - h(k(\theta)n(x)a(y)) \gg 1.$$

The result now follows immediately by integration by parts with respect to y.

6.2. **Contribution from θ near 0.**

We now prove the rapid decay of the second integral,

$$\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} b(y)b_2(\theta)\exp(i\lambda x + (1/2 - is)h(k(\theta)n(x)a(y)))d\theta dy.$$

We do this by estimating the sub-integral

$$\int_{-\pi}^{\pi} b_2(\theta)\exp(-ish(k(\theta)n(x)a(y)))d\theta,$$

where we have again absorbed the factor of $\exp(h(k(\theta)n(x)a(y))/2)$ into b_2, and the support of b and b_2 allow us to assume that $|y| \leq 10w$ and $|\theta| \leq 2c\kappa$. We shall do this with the aid of the following uniformizaton result for the function h, which is a special case of Propositions 13 or 14 of [13].

Lemma 16. *If $I \subset \mathbb{R}$ is a bounded open interval, there exists a neighbourhood $U \subset K$ of the identity and a real analytic function z on $U \times I$ such that $z(0, y) = 0$,*

$$h(k(\theta)a(y)) = y - yz^2,$$

and all derivatives of the change of co-ordinates from (θ, y) to (z, y) and back are bounded.

Let $r = r(x, y)$ be the hyperbolic distance from i to the point $n(x)a(y)$, and let α be the angle subtended at i between $n(x)a(y)$ and $i\infty$, so that

$$n(x)a(y)i = k(\alpha)a(r)i \in \mathbb{H}. \tag{25}$$

We have $r \gg x$, and our assumptions that $|y| \leq 10w$ and $x \leq C\kappa w \leq 2Cw$ imply that $r \ll w \ll 1$. Moreover, our assumptions that $x \sim \kappa w$ and $|y| \leq 10w$ imply that $\pi - 2c_1\kappa \geq \alpha \geq 2c_1\kappa$ for some absolute constant $c_1 > 0$, and hyperbolic trigonometry gives $r\alpha \sim x \sim w\kappa$.

Choose I in Lemma 16 to be any interval that contains $r(x, y)$ for all x and y under consideration, and let U be the open set produced by the Lemma. If we assume that α is small enough that $k(\alpha + \rho) \in U$ for $|\rho| \leq 2c\kappa$, Lemma 16 and (25) give

$$\begin{aligned} h(k(\theta)n(x)a(y)) &= h(k(\alpha+\theta)a(r)) \\ &= r - rz^2(\theta+\alpha, r) \end{aligned}$$

for $|\theta| \leq 2c\kappa$. We therefore have

$$\begin{aligned} \int_{-\infty}^{\infty} b_2(\theta)\exp(-ish(k(\theta)n(x)a(y)))d\theta &= e^{-irs}\int_{-\infty}^{\infty} b_2(\theta)\exp(-irsz^2(\theta+\alpha, r))d\theta \\ &= e^{-irs}\int_{-\infty}^{\infty} b_2(\theta-\alpha)\exp(-irsz^2(\theta, r))d\theta. \end{aligned} \tag{26}$$

Because $\operatorname{supp} b_2 \subset [-2c\kappa, 2c\kappa]$ and $\alpha \geq 2c_1\kappa$, we may assume that $b_2(\theta - \alpha)$ is supported outside of $[-c_1\alpha, c_1\alpha]$ by choosing c sufficiently small. Lemma 16 allows us to change variables from θ to z and rewrite (26) as

$$\int_{-\infty}^{\infty} \sigma_0(z)\exp(-irsz^2)dz,$$

where the bounds on the derivatives of the change of co-ordinates provided by the Lemma imply that σ_0 is a cutoff function at scale κ around α, and is again supported outside of some interval $[-c_2\alpha, c_2\alpha]$. We then change variable from z to κz to obtain

$$\int_{-\infty}^{\infty} \sigma_0(\kappa z)\exp(-irs\kappa^2z^2)\kappa dz = \int_{-\infty}^{\infty} \sigma_1(z)\exp(-irs\kappa\alpha(\kappa z^2/\alpha))\kappa dz,$$

where σ_1 is a cutoff function at scale 1 around α/κ whose support is bounded outside of $[-c_2\alpha/\kappa, c_2\alpha/\kappa]$. We finally change variable to $y = \kappa z^2/\alpha$, so that

$$z = \sqrt{\alpha y/\kappa}, \quad \frac{d^n z}{dy^n} \sim \sqrt{\alpha/\kappa}y^{1/2-n}.$$

On the support of σ_1 we have $y \geq c_2\alpha/\kappa$, so that

$$\left|\frac{d^n z}{dy^n}\right| \ll_n 1 \text{ for } n \geq 1,$$

and all derivatives of σ_1 with respect to y are bounded. With this final change of variables the integral becomes

$$\int_{-\infty}^{\infty} \sigma_2(y)\exp(-irs\kappa\alpha y)\kappa dy$$

where all derivatives of σ_2 are bounded, and this will be $\ll_A t^{-A}$ if $rs\kappa\alpha \gg t^\epsilon$. However, we have $r\alpha \sim x \sim \kappa w$, so $rs\kappa\alpha \gg t\kappa^2 w \gg t^\epsilon$ by our assumption on w.

We now suppose that α is not close to 0. If it is close to π, we may apply Lemma 16 again with $y \leq 0$ and argue as before. If α is bounded away from 0 and π, we may show that

$$(\partial/\partial\theta)h(k(\theta)n(x)a(y)) \gg r, \quad |(\partial/\partial\theta)^n h(k(\theta)n(x)a(y))| \ll_n r$$

for $|\theta| \leq 2c\kappa$, and the result now follows by integration by parts in θ provided $tr\kappa \gg t^\epsilon$. However, $tr\kappa \gg tx\kappa \sim t\kappa^2 w \gg t^\epsilon$ as required.

□

By inverting the Harish-Chandra transform and observing that $\widehat{k}_t$ decays rapidly in t outside of the intervals $[\pm t - t^\epsilon, \pm t + t^\epsilon]$, we obtain the following corollary of Proposition 14.

Corollary 17. *With notations as in Proposition 14, we have*

$$\left| \int_{-\infty}^{\infty} b(y) e^{i\lambda y} k_t(\rho(y)) dy \right| \ll_A t^{-A}.$$

7. Proof of Proposition 10

We now deduce Proposition 10 from Corollary 17. The first inequality (10) follows trivially from (6) of Proposition 8. The proof of inequality (11) splits into two cases depending on whether short extensions of ℓ and ℓ' intersect or not, and we begin with the intersecting case as it is the more difficult of the two.

Let $\widetilde{\ell}$ and $\widetilde{\ell}'$ be the segments obtained by extending ℓ and ℓ' by their length in each direction, and suppose that $\widetilde{\ell}$ and $\widetilde{\ell}'$ intersect. We shall change notation, and use ℓ and ℓ' to denote our new pair of extended intersecting geodesics whose length is now $3l \leq 3$. Let α be the angle at which ℓ and ℓ' meet, which is equal to κ up to absolutely bounded factors so that $\alpha \gg t^{-1/2+\epsilon}$. We choose length parametrizations ρ and ρ' of ℓ and ℓ' such that $\rho(0)$ and $\rho'(0)$ are their point of intersection, and extend ℓ and ℓ' further so that the domains of ρ and ρ' are $[-3l, 3l]$.

We begin by introducing functions which form a partition of unity on the support of the integrand in

$$I(t, \ell, \ell') = \int_{-3l}^{3l} \int_{-3l}^{3l} a(x_1)\phi(x_1)a(x_2)\overline{\phi(x_2)} K_t(\rho(x_1), \rho(x_2)) dx_1 dx_2.$$

Let b_1 and b_2 be positive C_0^∞ functions with support in $[-2, 2]$, with the property that $b_1(x_1/x_2)$ and $b_2(x_2/x_1)$ form a partition of unity on $\mathbb{RP}^1$. Let χ_1 be the characteristic function of $[-\kappa^{-2}t^{-1+\epsilon}, \kappa^{-2}t^{-1+\epsilon}]$, and χ_2 be the characteristic function of $[-3l, -\kappa^{-2}t^{-1+\epsilon}] \cup [\kappa^{-2}t^{-1+\epsilon}, 3l]$. We define

$$\begin{aligned} A_1(x_1, x_2) &= b_1(x_1/x_2)\chi_1(x_2), \\ A_2(x_1, x_2) &= b_2(x_2/x_1)\chi_1(x_1), \\ B_1(x_1, x_2) &= b_1(x_1/x_2)\chi_2(x_2), \\ B_2(x_1, x_2) &= b_2(x_2/x_1)\chi_2(x_1). \end{aligned}$$

It can be seen that $A_1 + A_2 + B_1 + B_2$ is a partition of unity on the support of $a(x_1)a(x_2)$, and that $A_1 + A_2$ is supported on the set $[-2\kappa^{-2}t^{-1+\epsilon}, 2\kappa^{-2}t^{-1+\epsilon}]^2$.

The contribution to $I(t, \ell, \ell')$ from $A_1 + A_2$ is bounded by a fixed multiple of

$$\int_{-2\kappa^{-2}t^{-1+\epsilon}}^{2\kappa^{-2}t^{-1+\epsilon}} \int_{-2\kappa^{-2}t^{-1+\epsilon}}^{2\kappa^{-2}t^{-1+\epsilon}} |\phi(x_1)\phi(x_2)K_t(\rho(x_1), \rho'(x_2))| dx_1 dx_2.$$

We shall estimate this integral in two ways. The first is by applying Cauchy-Schwarz to $\phi \times \overline{\phi}$ and K_t. The L^2 norm of $\phi \times \overline{\phi}$ is ≤ 1 by our L^2 normalization of ϕ, and the L^2 norm of K_t restricted to this region of $\ell \times \ell'$ may be estimated by $t^{1/2+\epsilon}(\kappa t^{1/2})^{-1}$ using (6) from Proposition 8, so that the integral is also $\ll t^{1/2+\epsilon}(\kappa t^{1/2})^{-1}$. The second way is by applying the bound $|\phi \times \overline{\phi}| \ll \beta^2$ and estimating the integral of K_t using the bound (6) of Proposition 8, which gives a bound for the integral of $\beta^2 t^{1/2+\epsilon}(\kappa t^{1/2})^{-3}$.

As a result of this, it suffices to show that the integral I_1 defined by

$$I_1 = \int_{-3l}^{3l} \int_{-3l}^{3l} B_1(x_1, x_2)a(x_1)\phi(x_1)a(x_2)\overline{\phi(x_2)}K_t(\rho(x_1), \rho'(x_2))dx_1 dx_2$$

is rapidly decaying, and likewise for the contribution from B_2. We shall do this by expanding ϕ in its Fourier series and proving the estimate

$$\left| \int_{-3l}^{3l} b(y/x_2)a(y)e^{i\lambda y}K_t(\rho(y), \rho'(x_2))dy \right| \ll_A t^{-A} \tag{27}$$

for the integrals in x_1 with $x_2 \in [-3l, -\kappa^{-2}t^{-1+\epsilon}] \cup [\kappa^{-2}t^{-1+\epsilon}, 3l]$ held fixed, uniformly in x_2. We shall assume that $x_2 \in [\kappa^{-2}t^{-1+\epsilon}, 3l]$, as the other case may be treated in the same way.

Applying a hyperbolic isometry, we may assume without loss of generality that $\rho'(x_2)$ is the point $i \in \mathbb{H}$, and that ℓ is a segment of a vertical geodesic lying to the right of i. Our assumptions that the support of a vanished near the endpoints of ℓ, and that the support of k_t was small, imply that if the standard horocycle $n(x)i$ through $\rho'(x_2)$ does not meet ℓ then the LHS of (27) is 0. We may therefore let this point of intersection be $\rho(q)$, where q satisfies $|q| \leq x_2$ by geometric considerations. We let x be the unique number such that $n(x)i \in \ell$, so that $\rho(y) = n(x)a(y - q)$. Hyperbolic trigonometry and our observation that $\kappa \sim \alpha$ gives the existence of an absolute constant C such that $C^{-1}\kappa x_2 \leq x \leq C\kappa x_2$.

If we change our co-ordinate in the LHS of (27) to $z = y - q$, so that the parametrisation of ℓ is now $\rho(z + q) = n(x)a(y)$, the integral becomes

$$\begin{aligned} e^{-i\lambda q} \int_{-\infty}^{\infty} b_1 \left(\frac{z+q}{x_2} \right) a(z+q)e^{i\lambda x}K_t(\rho(z+q), \rho'(x_2))dz \\ = e^{-i\lambda q} \int_{-\infty}^{\infty} b_1 \left(\frac{z+q}{x_2} \right) a(z+q)e^{i\lambda x}k_t(\rho(z+q))dz. \end{aligned} \tag{28}$$

Because $|q| \leq x_2$, we have the bounds

$$\begin{aligned} \operatorname{supp} b_1\left(\frac{z+q}{x_2}\right) &\subseteq [-10x_2, 10x_2] \\ \left|\frac{d^n}{dz^n} b_1\left(\frac{z+q}{x_2}\right)\right| &\ll_n x_2^{-n}. \end{aligned}$$

We assumed that $\kappa^{-2}t^{-1+\epsilon} \leq x_2 \leq 3l \leq 3$, and know that $C^{-1}\kappa x_2 \leq x \leq C\kappa x_2$ for some absolute constant $C > 0$ and that the function

$$b_1\left(\frac{z+q}{x_2}\right) a(z+q)$$

satisfies (19) and (20) (with x_2 in place of w). The integral (28) therefore satisfies the assumptions of Corollary 17 with $w = x_2$, and so is $\ll_A t^{-A}$.

In the case where the extensions of ℓ and ℓ' do not intersect, we see that the distance from any point of ℓ to ℓ' is $\gg \kappa$. We may therefore apply Corollary 17 with $w = 1$ to show that the sub-integrals with respect to x_2 in $I(t, \ell, \ell')$ are $\ll t^{-A}$, which gives the stronger estimate $|I(t, \ell, \ell')| \ll_A t^{-A}$. This completes the proof of Proposition 10.

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